On the universal constraints for relaxation rates for quantum dynamical semigroup

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in collaboration with
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Andrzej Kossakowski (1938-2021)

Markovian semigroups

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ON QUANTUM STATISTICAL MECHANICS OF NON-HAMILTONIAN SYSTEMS

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An axiomatic definition of time evolution (dynamical semi-group) of a physical system has been given. A dynamical semi-group is defined as a one-parameter semi-group of linear endomorphisms of the set of all density operators corresponding to the physical system in question. Some classes of dynamical semi-groups (quantum Poisson and Brownian processes) induced by Markov processes on topological groups are described. Examples of dynamical semi-groups for the harmonic oscillator are given.

Markovian semigroups

THEORETICAL PHYSICS

On Necessary and Sufficient Conditions for a Generator of a Quantum Dynamical Semi-Group

by

A. KOSSAKOWSKI

Presented by W. RUBINOWICZ on June 14, 1972

Summary. A dynamical semi-group has been defined as a one-parameter contracting semi-group of trace preserving linear operators on the real Banach space $L^1(\mathcal{H})$ of self-adjoint trace class linear operators on a separable complex Hilbert space $\mathcal{H}$. It has been proved that a linear operator L with the domain $D(L)$ and the range $R(L)$ both in $L^1(\mathcal{H})$ generates dynamical semi-group $\mathcal{S}(\mathcal{H})=\{S_t; t \geq 0\}$ iff the domain $D(L)$ is dense in $L^1(\mathcal{H})$, $R(I-L)=L^1(\mathcal{H})$, L is a dissipative operator in the sense of Lumer and Phillips, and $\mathrm{Tr}(L\rho)=0$ for all $\rho \in D(L)$. The resulting master equation $d/dt\,(S_t\,\rho)=L\,(S_t\,\rho)$, $\rho \in D(L)$, satisfies the positivity and normalization requirements.

Completely positive dynamical semigroups of N-level systems*

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(Received 19 March 1975)

We establish the general form of the generator of a completely positive dynamical semigroup of an N-level quantum system, and we apply the result to derive explicit inequalities among the physical parameters characterizing the Markovian evolution of a 2-level system.

I. INTRODUCTION

the gme does indeed go over into

GKS 1976

Theorem 2.2. A linear operator $L: M(N) \to M(N)$ is the generator of a completely positive dynamical semigroup of $M(N)$ if it can be expressed in the form

$$L: \rho \to L\rho = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij} \{[F_i, \rho F_j^*] + [F_i \rho, F_j^*]\}, \quad \rho \in M(N), \tag{2.3}$$

Ingarden-Kossakowski-Sudarshan-Gorini

Markovian semigroup

$$\dim \mathcal{H} = d < \infty \; ; \quad \hbar = 1$$

$$\frac{d}{dt}\Lambda_t = \mathcal{L}\Lambda_t$$

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{t\mathcal{L}}$ *is CPTP for* $t \geq 0$ *if and only if*

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

$$\Phi(\rho) := \sum_k \gamma_k L_k \rho L_k^\dagger \quad \text{(Completely positive)}$$

$$(\Phi^\ddagger(X), \rho) = (X, \Phi(\rho)) \quad \text{(HS inner product)}$$

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \Phi(\rho) - \frac{1}{2}\{\Phi^\ddagger(\mathbb{1}), \rho\}$$

Schrödinger vs. Heisenberg

Λ_t is CPTP $\longleftrightarrow$ $\Lambda_t^\ddagger$ is CP unital

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \Phi(\rho) - \frac{1}{2}\{\Phi^\ddagger(\mathbb{1}),\rho\} \;\longrightarrow\; \Lambda_t = e^{t\mathcal{L}}$$

$$\mathrm{Tr}\,\mathcal{L}(\rho) = 0 \;\longleftrightarrow\; \mathrm{Tr}\,\Lambda_t(\rho) = \mathrm{Tr}\,\rho$$

$$\mathcal{L}^\ddagger(X) \;=\; i[H,X] + \Phi^\ddagger(X) - \frac{1}{2}\{\Phi^\ddagger(\mathbb{1}),X\} \;\longrightarrow\; \Lambda_t^\ddagger = e^{t\mathcal{L}^\ddagger}$$

$$\mathcal{L}^\ddagger(\mathbb{1}) = 0 \;\longleftrightarrow\; \Lambda_t^\ddagger(\mathbb{1}) = \mathbb{1}$$

Given $\Phi : \mathcal{M}_d \to \mathcal{M}_d$, it is CP iff

$$(\mathrm{id} \otimes \Phi)\mathbf{P} \geq 0$$

$$\mathbf{P} = \frac{1}{d}|\psi\rangle\langle\psi| \; ; \quad \psi = \sum_k |k \otimes k\rangle$$

Given $\mathcal{L} : \mathcal{M}_d \to \mathcal{M}_d$, it generates CP semigroup iff

$$\mathbf{P}^{\perp}[(\mathrm{id} \otimes \mathcal{L})\mathbf{P}]\mathbf{P}^{\perp} \geq 0$$

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

- γ_k are not directly measured in the lab.
- they DO depend on the representation

Which physical quantities decide about complete positivity?

Spectrum

$$\mathcal{L}(X_\alpha) = \ell_\alpha X_\alpha \quad (\Rightarrow \quad \mathcal{L}(X_\alpha^\dagger) = \ell_\alpha^* X_\alpha^\dagger)$$

Theorem

- $\ell \in \mathrm{spec}(\mathcal{L}) \;\; \Leftrightarrow \;\; \ell^* \in \mathrm{spec}(\mathcal{L})$
- *there is a leading eigenvalue* $\ell_0 = 0$
- *the corresponding eigenvector defines a density operator*
- $\mathrm{Re}\, \ell_\alpha \leq 0$ *for* $\alpha = 1, 2, \ldots, d^2 - 1$.

$$\Gamma_\alpha := -\mathrm{Re}\, \ell_\alpha \quad \text{(relaxation rates)}$$

$$T_\alpha := 1/\Gamma_\alpha \quad \text{(relaxation times)}$$

γ_α vs Γ_α

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

Positivity of Γ_α is necessary but NOT sufficient for CP

Example: qubit

$$\mathcal{L}(\rho) = \frac{1}{2}\sum_{k=1}^{3} \gamma_k (\sigma_k \rho \sigma_k - \rho)$$

$$\Gamma_1 = \gamma_2 + \gamma_3 \ , \ \Gamma_2 = \gamma_3 + \gamma_1 \ , \ \Gamma_3 = \gamma_1 + \gamma_2$$

$$\rho = \frac{1}{2}\Big(\mathbb{1} + \sum_k x_k \sigma_k\Big)$$

$$\mathbf{x} = (x_1, x_2, x_3) \ \rightarrow \ \mathbf{x}(t) = \left(e^{-\Gamma_1 t} x_1, e^{-\Gamma_2 t} x_2, e^{-\Gamma_3 t} x_3\right)$$

$$\mathbf{x}(t) \in \mathbf{B} \iff \Gamma_k \geq 0 \ \ (k = 1, 2, 3)$$

NOT enough for complete positivity!

Example: qubit

$$\mathcal{L}(\rho) = \frac{1}{2}\sum_{k=1}^{3} \gamma_k(\sigma_k \rho \sigma_k - \rho)$$

$$\Gamma_1 = \gamma_2 + \gamma_3\ ,\ \Gamma_2 = \gamma_3 + \gamma_1\ ,\ \Gamma_3 = \gamma_1 + \gamma_2$$

$\mathcal{L}$ is a legitimate Lindbladian $\iff \gamma_k \geq 0 \iff$

$$\Gamma_1 \leq \Gamma_2 + \Gamma_3\ , \quad \Gamma_2 \leq \Gamma_3 + \Gamma_1\ , \quad \Gamma_3 \leq \Gamma_1 + \Gamma_2$$

$$\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$$

$$\Gamma_k \leq \frac{1}{2}\Gamma \quad (k = 1, 2, 3)$$

Example: GKS 1976

$$\mathcal{L}(\rho) = -i\frac{\Delta}{2}[\sigma_z, \rho] + \gamma_+\mathcal{L}_+ + \gamma_-\mathcal{L}_- + \gamma_z\mathcal{L}_z$$

$$\begin{aligned}
\mathcal{L}_+(\rho) &= \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\} \\
\mathcal{L}_-(\rho) &= \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\} \\
\mathcal{L}_z(\rho) &= \sigma_z\rho\sigma_z - \rho
\end{aligned}$$

$$\Gamma_{\mathrm{T}} := \Gamma_1 = \Gamma_2 = \frac{1}{2}(\gamma_+ + \gamma_-) + 2\gamma_z \ ; \quad \Gamma_{\mathrm{L}} = \Gamma_3 = \gamma_+ + \gamma_-$$

Example: GKS 1976

$$\Gamma_{\mathrm{T}} := \Gamma_1 = \Gamma_2 = \frac{1}{2}(\gamma_+ + \gamma_-) + 2\gamma_z \; ; \quad \Gamma_{\mathrm{L}} = \Gamma_3 = \gamma_+ + \gamma_-$$

$$\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3 = 2(\gamma_+ + \gamma_- + \gamma_z)$$

$$\Gamma_k \leq \frac{1}{2}\Gamma \quad (k = 1, 2, 3)$$

Bloch equations

$$M_k := \mathrm{Tr}(\rho\sigma_k)$$

$$\begin{aligned}\dot{M}_x &= \Delta M_y - \frac{M_x}{T_\mathrm{T}} \\ \dot{M}_y &= -\Delta M_x - \frac{M_y}{T_\mathrm{T}} \\ \dot{M}_z &= -\frac{M_z - M_0}{T_\mathrm{L}}\end{aligned}$$

$$2\,T_\mathrm{L} \geq T_\mathrm{T}$$

well tested!

General qubit Lindbladian

Theorem (G. Kimura, PRA 2002)

For any qubit Lindblad generator

$$\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$$

$$\Gamma_k \leq \frac{1}{2}\Gamma \quad (k = 1, 2, 3)$$

1. $\ell_1 = \ell_2^* = -\Gamma_T + i\omega \ , \quad \ell_3 = -\Gamma_L,$
2. $\ell_k = -\Gamma_k, \quad k = 1, 2, 3$

$$\Gamma_1 + \Gamma_2 \geq \Gamma_3 \ , \quad \text{etc}$$

general d

Wolf & Cirac, CMP (2008): $\mathcal{L} = \mathcal{L}_D$

$$\|\mathcal{L}\|_\infty \leq \frac{2}{d}\,\Gamma$$

$$\Gamma_k = -\mathrm{Re}\,\ell_k \implies \Gamma_k \leq |\ell_k| \leq \|\mathcal{L}\|_\infty \leq \frac{2}{d}\,\Gamma$$

Kimura et al, OSID (2017): arbitrary $\mathcal{L}$

$$\Gamma_k \leq \frac{\sqrt{2}}{d}\,\Gamma$$

Conjecture: $\Gamma_k \leq \frac{1}{d}\,\Gamma \qquad (d > 2)$

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Conjecture: $\Gamma_k \leq \frac{1}{d}\,\Gamma \qquad (d > 2)$

Conjecture

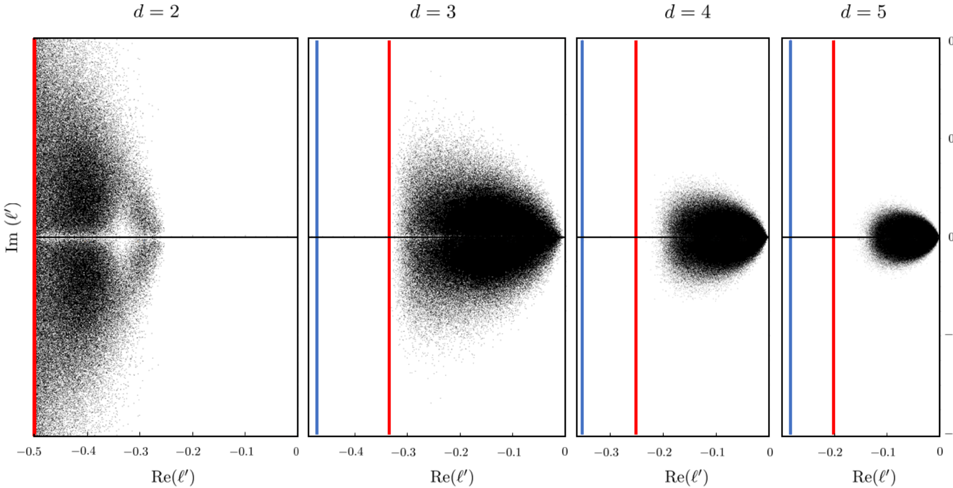
d-level Lindbladian

$$\Gamma := \Gamma_1 + \ldots + \Gamma_{d^2-1}$$

$$\Gamma_k \leq \frac{1}{d}\Gamma \quad (k = 1, \ldots, d^2 - 1)$$

$$R_k := \frac{\Gamma_k}{\Gamma}$$

$$R_k \leq \frac{1}{d} \quad (k = 1, \ldots, d^2 - 1)$$



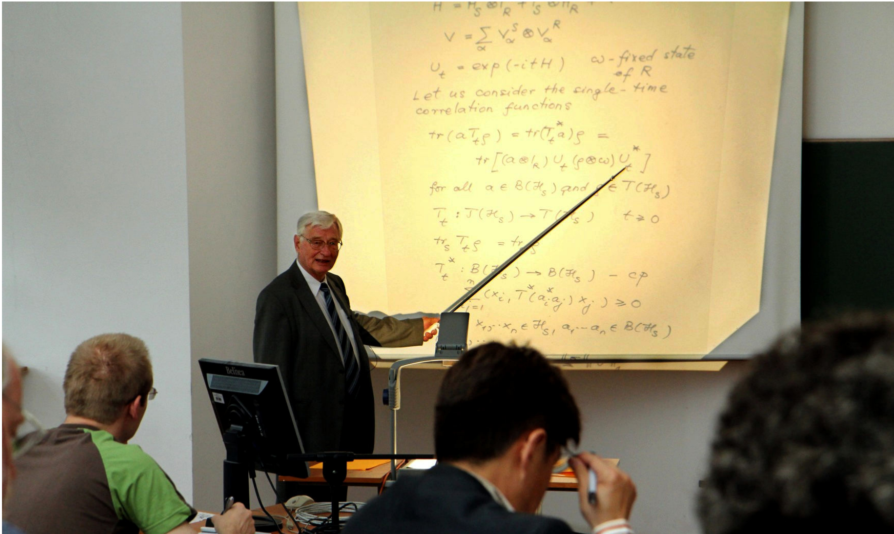
Distribution of ℓ_α/Γ for random Lindbladians. Red vertical lines denote the bound '$-1/d$'. Blue vertical lines denote the bound '$-\sqrt{2}/d$'.

S. Denisov, T. Laptyeva, W. Tarnowski, DC, and K. Życzkowski, Universal Spectra of Random Lindblad Operators, PRL **123**, 140403 (2019).

$$\Gamma_k \leq \frac{1}{d}\Gamma$$

- is tight
- **true** for 'classical' generator
- **true** for unital semigroups
- **true** for Davies generators
- implications

44 SMP (Toruń 2012)



40 years after (Toruń 2016)

The bound cannot be improved

$$\Gamma_k \leq \frac{1}{d}\,\Gamma$$

$$\Sigma^\dagger = \Sigma = \sum_k s_k |k\rangle\langle k| \ ; \quad s_1 \leq \ldots \leq s_d$$

$$\mathcal{L}(\rho) = \Sigma\rho\Sigma - \frac{1}{2}\{\Sigma^2, \rho\} = -\frac{1}{2}[\Sigma, [\Sigma, \rho]]$$

$$\Gamma_{ij} = (s_i - s_j)^2$$

$$s_2 = \ldots = s_{d-1} = \frac{s_1 + s_d}{2} \implies \Gamma_{\max} = \frac{1}{d}\,\Gamma$$

Classical generator

$$\mathbf{T}(t) = e^{t\mathbf{K}} = \text{semigroup of stochastic matrices}$$

$$\mathbf{K}_{ij} \geq 0 \ (i \neq j) \ ; \quad \sum_{i=1}^{d} \mathbf{K}_{ij} = 1$$

$$\mathbf{K}_{ij} = t_{ij} - \delta_{ij} \sum_{k=1}^{d} t_{kj} \ ; \quad t_{ij} \geq 0$$

Pauli Master Equation: $\dot{p}_i = \sum_j \Big(t_{ij} p_j - t_{ji} p_i \Big)$

$$\Gamma_k^{\rm cl} = -{\rm Re}\, \ell_k^{\rm cl} \geq 0 \quad (k = 1, \ldots, d-1)$$

$\Gamma_k^{\rm cl}$ can be completely arbitrary

$\mathbf{K}_{ij} = t_{ij} - \delta_{ij} \sum_k t_{kj}$; $t_{ij} \geq 0$

$$\mathcal{L}(\rho) = \sum_{i,j=1}^{d} t_{ij} \mathrm{L}_{ij} \rho \mathrm{L}_{ij}^{\dagger} - \frac{1}{2}\{B, \rho\} \quad ; \; \mathrm{L}_{ij} = |i\rangle\langle j|$$

$$B = \sum_k b_k |k\rangle\langle k| \; ; \quad b_k = \sum_{j=1}^{d} t_{jk}$$

$$\left\{\Gamma_1^{\mathrm{cl}}, \ldots, \Gamma_{d-1}^{\mathrm{cl}}\right\} \cup \left\{\Gamma_{ij} = \frac{1}{2}(b_i + b_j) \; ; (i \neq j)\right\}$$

Theorem

$$\Gamma := \sum_k \Gamma_k^{\mathrm{cl}} + \sum_{i \neq j} \Gamma_{ij}$$

$$\Gamma_k^{\mathrm{cl}} \leq \frac{1}{d} \Gamma \qquad \& \qquad \Gamma_{ij} \leq \frac{1}{d} \Gamma$$

Schrödinger vs. Heisenberg

$$\mathcal{L}(X_\alpha) = \ell_\alpha X_\alpha$$

$$\mathcal{L}^\ddagger(Y_\alpha) = \ell_\alpha^* Y_\alpha$$

$$\mathcal{L}(\omega) = 0$$

$$\mathcal{L}^\ddagger(\mathbb{1}) = 0$$

ω = invariant state

$$\mathcal{L}^{\ddagger}(Y_\alpha) = \ell_\alpha^* Y_\alpha$$

$$\omega = \text{invariant state} \quad \longrightarrow \quad \mathcal{L}(\omega) = 0$$

$$(A, B)_\omega := \mathrm{Tr}(\omega A^\dagger B) \quad \longrightarrow \quad \|A\|_\omega^2 = (A, A)_\omega$$

Theorem

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|_\omega^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|_\omega^2$$

Unital semigroup

$$\omega = \frac{1}{d}\,\mathbb{1}$$

$$\mathcal{L}(\mathbb{1}) = 0$$

$$\Lambda_t := e^{t\mathcal{L}} \quad \text{is unital iff} \quad \frac{d}{dt}S(\Lambda_t\rho) \geq 0$$

Unital semigroup

$$\omega = \frac{1}{d}\,\mathbb{1}$$

$$\mathcal{L}(\mathbb{1}) = 0$$

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|_\omega^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|_\omega^2$$

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2 \quad ; \quad \|A\|^2 = \mathrm{Tr}(A^\dagger A)$$

Unital semigroup

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2$$

Bötcher & Wenzel, Lin. Alg. Appl. (2008)

$$\|[A,B]\|^2 \leq 2\|A\|^2\|B\|^2$$

$$\Gamma_\alpha \leq \sum_k \gamma_k \|L_k\|^2$$

Unital semigroup

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \quad ; \quad \gamma_k > 0$$

$$\Gamma_\alpha \leq \sum_k \gamma_k \|L_k\|^2$$

$$\|L_k\|^2 = 1 \quad \longrightarrow \quad \sum_k \gamma_k = \frac{1}{d} \sum_\alpha \Gamma_\alpha = \frac{1}{d}\,\Gamma$$

Theorem

For a unital semigroup

$$\Gamma_\alpha \leq \frac{1}{d}\,\Gamma$$

Beyond unital semigroup — covariant generators

$$U_{\mathbf{x}}\mathcal{L}(X)U_{\mathbf{x}}^{\dagger} = \mathcal{L}(U_{\mathbf{x}}XU_{\mathbf{x}}^{\dagger})$$

$$U_{\mathbf{x}} = \sum_{k=1}^{d} e^{-ix_k}|k\rangle\langle k| \quad ; \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$$

Beyond unital semigroup — covariant generators

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_0(\rho) = -i[H,\rho] \; ; \quad H = \sum_i h_i |i\rangle\langle i|$$

$$\mathcal{L}_1(\rho) = \sum_{i\neq j=1}^{d} t_{ij} \left(L_{ij}\rho L_{ij}^\dagger - \frac{1}{2}\delta_{ij}\{|j\rangle\langle j|, \rho\} \right)$$

$$\mathcal{L}_2(\rho) = \sum_{i,j=1}^{d} d_{ij} \left(|i\rangle\langle i|\rho|j\rangle\langle j| - \frac{1}{2}\delta_{ij}\{|j\rangle\langle j|, \rho\} \right)$$

$$t_{ij} \geq 0 \;, \quad [d_{ij}] \geq 0 \; ; \quad L_{ij} = |i\rangle\langle j|$$

$$\Gamma_\alpha \leq \frac{1}{d}\,\Gamma$$

Weak coupling limit — Davies generators

$$\mathcal{L}(\rho) = -i[H,\rho] + \mathcal{L}_D(\rho)$$

- $H = \sum_i h_i |i\rangle\langle i|$
- $\omega = \sum_i p_i |i\rangle\langle i|$
- $\mathcal{L}_D^{\ddagger}$ is Hermitian w.r.t. $(\ ,\ )_\omega$

$$U_{\mathbf{x}} \mathcal{L}(X) U_{\mathbf{x}}^\dagger = \mathcal{L}(U_{\mathbf{x}} X U_{\mathbf{x}}^\dagger)$$

$$\Gamma_\alpha \leq \frac{1}{d}\Gamma$$

Implications

$$\Gamma_\alpha \leq \frac{1}{d}\Gamma$$

- spectra of channels
- is quantum channel Φ Markovian, that is, $\Phi = e^{\mathcal{L}}$?

Spectra of channels: $\Gamma_\alpha \leq \frac{1}{d}\Gamma$

$$\mathcal{L} = \Phi - \mathrm{id}$$

Theorem

The spectrum $z_\alpha = x_\alpha + iy_\alpha$ *of any unital quantum channel satisfy*

$$\sum_{\beta=1}^{d^2-1} x_\beta \leq d(d-1) - 1 + dx_\alpha,$$

for $\alpha = 1, \dots, d^2 - 1$.

Theorem

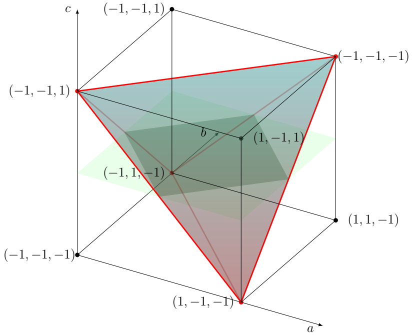
The spectrum $z_\alpha = x_\alpha + iy_\alpha$ *of any qubit channel satisfies*

$$|x_1 \pm x_2| \leq 1 \pm x_3.$$

Pauli qubit channels: $\Phi(\rho) = \sum_{\alpha=1}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha$

$$\widehat{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad ; \quad \Delta = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$$

$$\textbf{Algoet – Fujiwara :} \quad |\lambda_1 \pm \lambda_2| \leq 1 \pm \lambda_3 \longleftrightarrow |x_1 \pm x_2| \leq 1 \pm x_3$$



Markovianity $\Gamma_\alpha \leq \frac{1}{d}\Gamma$

M. Wolf, et al PRL (2008).
Unital Φ. Is $\Phi = e^{\mathcal{L}}$?

$$\mathcal{L} \quad \longrightarrow \quad \ell_\alpha = iy_\alpha - \Gamma_\alpha$$

$$\Phi \quad \longrightarrow \quad z_\alpha = e^{\ell_\alpha}$$

$$\det \Phi = z_1 \ldots z_{d^2-1} = e^{-\Gamma} \leq e^{d\Gamma_\alpha} = |z_\alpha|^d$$

$$\sqrt[d]{\det \Phi} \leq |z_\alpha| \leq 1$$

Frobenius-Perron ring

Example: qubit Pauli channel

$$z_k = e^{-\Gamma_k}$$

$$z_1 z_2 z_3 \leq z_k^2 \quad (k = 1, 2, 3)$$

$$z_1 z_2 \leq z_3 \ , \quad \text{etc.}$$

Davalos et al, Quantum (2019)
Puchała et al, Phys. Lett. A (2019)

Conlusions

- For any qubit Lindbladian $\Gamma_k \leq \frac{1}{2}\Gamma$
- For any unital Lindbladian $\Gamma_k \leq \frac{1}{d}\Gamma$
- True for any 'physical generator' obtained in the weak coupling limit
- The conjecture for arbitrary $\mathcal{L}$ is supported by numerical analysis
- New bounds for the spectra of channels
- Necessary condition for Markovianity (for unital channels) $\sqrt[d]{\det \Phi} \leq |z_\alpha| \leq 1$
- D.C., G. Kimura, A. Kossakowski, and Y. Shishido, *On the universal constraints for relaxation rates for quantum dynamical semigroup*, arxiv:2011.10159